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Counter-rotating relativistic static thin disks

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Abstract

A detailed study of the Counter-Rotating Model (CRM) for generic finite static axially symmetric thin disks with nonzero radial pressure is presented. We find a general constraint over the counter-rotating tangential velocities needed to cast the surface energy-momentum tensor of the disk as the superposition of two counter-rotating perfect fluids. We also found expressions for the energy density and pressure of the counter-rotating fluids. Then we shown that, in general, it is not possible to take the two counter-rotating fluids as circulating along geodesics neither take the two counter-rotating tangential velocities as equal and opposite. An specific example is studied where we obtain some CRM with well defined counter-rotating tangential velocities that are agree with the strong energy condition, but there are regions of the disk with negative energy density, in violation of the weak energy condition.

1 Introduction

The study of axially symmetric solutions of Einstein field equations corresponding to disklike configurations of matter has a long history. These were first studied by Bonnor and Sackfield [1], obtaining pressureless static disks, and by Morgan and Morgan, obtaining static disks with and without radial pressure [2, 3]. In connection with gravitational collapse, disks were first studied by Chamorro, Gregory and Stewart [4]. In the last years, disks models

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with radial tension [5], magnetic fields [6] and magnetic and electric fields [7] have been also studied. Several classes of exact solutions of the Einstein field equations corresponding to static and stationary thin disks have been obtained by different authors [8 – 18], with or without radial pressure.

In the case of static disks without radial pressure, there are two common interpretations. The stability of these models can be explained by assuming either the existence of hoop stresses or that the particles on the disk plane move under the action of their own gravitational field in such a way that as many particles move clockwise as counterclockwise. This last interpretation, the “Counter-Rotating Model” (CRM), is frequently made since it can be invoked to mimic true rotational effects. Even though this interpretation can be seen as a device, there are observational evidence of disks made of streams of rotating and counter-rotating matter [19, 20].

Usually has been considered that the CRM can be applied only when we do not have radial pressure and the azimuthal stress is positive (pressure). These conditions, however, are very restrictive and, in many cases, we have disks models that only agree with them in a partial region. Thus, the CRM will be valid only as a partial interpretation of the corresponding disks. Another, common, assumption is to take the CRM as representing two fluids that circulate in opposite directions with the same tangential velocity. As we will show in this paper, this is not the case and, in general, the two fluids circulate with different velocities. Furthermore, in some cases may not be possible to obtain a CRM if the two tangential velocities are taken as equal and opposite. Also, commonly is assumed that the two counter-rotating fluids must be taken as circulating along geodesics. We also will show that this is not necessary and that only can be made if the radial pressure is constant.

The aim of this paper is a detailed study of the CRM for generic finite static axially symmetric thin disks with nonzero radial pressure. In section 2 we present a summary of the procedure to obtain these thin disks models and obtain the surface energy-momentum tensor of the disk. In the next section, section 3, we consider the CRM for the disk. We find a general constraint over the counter-rotating tangential velocities needed to cast the surface energy-momentum tensor of the disk as the superposition of two counter-rotating perfect fluids. We also found expressions for the energy density and pressure of the counter-rotating fluids. Then we shown that, in general, there is not possible to take the two counter-rotating tangential velocities as equal and opposite neither take the two counter-rotating fluids as circulating along geodesics. In section 4, we consider an specific example where we obtain some CRM with well defined counter-rotating tangential velocities. The CRM ob-

tained are agree with the strong energy condition, but there are regions of the disks with negative energy density, in violation of the weak energy condition. Finally, in section 5, we summarize our main results.

2 Relativistic Static Thin Disks

In this section we present, following closely reference [5], a summary of the procedure to obtain finite static axially symmetric thin disks with nonzero radial pressure. The metric can be written as the line element,

$$ds^2 = e^{-2\Phi}[\mathcal{R}^2 d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)] - e^{2\Phi} dt^2, \quad (2.1)$$

where Φ , Λ and \mathcal{R} are functions of r and z only. The Einstein vacuum equations for this metric are equivalent to the system

$$\mathcal{R}_{,rr} + \mathcal{R}_{,zz} = 0, \quad (2.2a)$$

$$(\mathcal{R}\Phi_{,r})_{,r} + (\mathcal{R}\Phi_{,z})_{,z} = 0, \quad (2.2b)$$

$$\mathcal{R}_{,z}\Lambda_{,r} + \mathcal{R}_{,r}\Lambda_{,z} - 2\mathcal{R}\Phi_{,r}\Phi_{,z} - \mathcal{R}_{,rz} = 0, \quad (2.2c)$$

$$\mathcal{R}_{,r}\Lambda_{,r} - \mathcal{R}_{,z}\Lambda_{,z} - \mathcal{R}(\Phi_{,r}^2 - \Phi_{,z}^2) + \mathcal{R}_{,zz} = 0, \quad (2.2d)$$

where we assume the existence of the second derivatives of the functions Φ , Λ and \mathcal{R} .

A general solution of the above system can be obtained by the following procedure. Let $\nu = r + iz$ and $\mathcal{F}(\nu)$ any analytical function of ν . Then we take

$$\mathcal{R}(r, z) = \text{Re } \mathcal{F}(\nu), \quad (2.3a)$$

$$\mathcal{Z}(r, z) = \text{Im } \mathcal{F}(\nu), \quad (2.3b)$$

$$\Phi(r, z) = \Psi(\mathcal{R}, \mathcal{Z}), \quad (2.3c)$$

$$\Lambda(r, z) = \Pi(\mathcal{R}, \mathcal{Z}) + \ln |\mathcal{F}'(\nu)|, \quad (2.3d)$$

where $\Psi(\mathcal{R}, \mathcal{Z})$ and $\Pi(\mathcal{R}, \mathcal{Z})$ are solutions of the Weyl equations [21, 22]

$$(\mathcal{R}\Psi)_{,\mathcal{R}} + (\mathcal{R}\Psi)_{,\mathcal{Z}} = 0 , \quad (2.4a)$$

$$\Pi_{,\mathcal{R}} = \mathcal{R}(\Psi_{,\mathcal{R}}^2 - \Psi_{,\mathcal{Z}}^2) , \quad (2.4b)$$

$$\Pi_{,\mathcal{Z}} = 2\mathcal{R}\Psi_{,\mathcal{R}}\Psi_{,\mathcal{Z}} . \quad (2.4c)$$

It is easy to see that the condition of integrability of the system (2.4b) – (2.4c) is guaranteed by the equation (2.4a). Also we can see that this equation is equivalent with the Laplace equation in flat three-dimensional space for an axially symmetric function and so Ψ can be taken as a solution for an appropriated Newtonian source with axial symmetry. Once a solution Ψ is known, Π is computed from (2.4b) – (2.4c) and so we obtain, from (2.3a) – (2.3d), a solution of the field equations (2.2a) – (2.2d).

Now if we assume that \mathcal{R} , Φ and Λ are symmetrical functions of z and that the first derivatives of the metric tensor are not continuous on the plane $z = 0$, with discontinuity functions

$$b_{ab} = g_{ab,z}|_{z=0+} - g_{ab,z}|_{z=0-} = 2 g_{ab,z}|_{z=0+} ,$$

the Einstein equations yield an energy-momentum tensor $T_a^b = Q_a^b \delta(z)$, where $\delta(z)$ is the usual Dirac function with support on the disk and

$$Q_b^a = \frac{1}{2} \{ b^{az} \delta_b^z - b^{zz} \delta_b^a + g^{az} b_b^z - g^{zz} b_b^a + b_c^c (g^{zz} \delta_b^a - g^{az} \delta_b^z) \}$$

is the distributional energy-momentum tensor. The “true” surface energy-momentum tensor (SEMT) of the disk, S_a^b , can be obtained through the relation

$$S_a^b = \int T_a^b ds_n = e^{\Lambda-\Phi} Q_a^b , \quad (2.5)$$

where $ds_n = \sqrt{g_{zz}} dz$ is the “physical measure” of length in the normal to the disk direction. For the metric (2.1) we obtain

$$S_0^0 = 2e^{\Phi-\Lambda} \left\{ \Lambda_{,z} - 2\Phi_{,z} + \frac{\mathcal{R}_{,z}}{\mathcal{R}} \right\} , \quad (2.6a)$$

$$S_1^1 = 2e^{\Phi-\Lambda} \Lambda_{,z} , \quad (2.6b)$$

$$S_2^2 = 2e^{\Phi-\Lambda} \left\{ \frac{\mathcal{R}_{,z}}{\mathcal{R}} \right\} , \quad (2.6c)$$

where all the quantities are evaluated at $z = 0^+$.

We can write the metric and the SEMT in the canonical forms

$$g_{ab} = -V_a V_b + W_a W_b + X_a X_b + Y_a Y_b , \quad (2.7a)$$

$$S_{ab} = \sigma V_a V_b + p_\varphi W_a W_b + p_r X_a X_b , \quad (2.7b)$$

with an orthonormal tetrad $e_{\hat{a}}^b = \{V^b, W^b, X^b, Y^b\}$, where

$$V^a = e^{-\Phi} (1, 0, 0, 0) , \quad (2.8a)$$

$$W^a = \frac{e^\Phi}{\mathcal{R}} (0, 1, 0, 0) , \quad (2.8b)$$

$$X^a = e^{\Phi-\Lambda} (0, 0, 1, 0) , \quad (2.8c)$$

$$Y^a = e^{\Phi-\Lambda} (0, 0, 1, 0) . \quad (2.8d)$$

The energy density, the azimuthal pressure, and the radial pressure are, respectively,

$$\sigma = -S_0^0 , \quad p_\varphi = S_1^1 , \quad p_r = S_2^2 , \quad (2.9)$$

and

$$\varrho = \sigma + p_\varphi + p_r \quad (2.10)$$

is the effective Newtonian density.

3 The Counter-Rotating Model

Now we consider that the SEMT S^{ab} can be written as the superposition of two counter-rotating perfect fluids that circulate in opposite directions; that is, based on the two-perfect-fluid model of anisotropic fluids [23], we assume that S^{ab} can be cast as

$$S^{ab} = S_+^{ab} + S_-^{ab} , \quad (3.1)$$

where S_+^{ab} and S_-^{ab} are, respectively, the SEMT of the prograd and retrograd counter-rotating fluids.

Let be $U_\pm^a = (U_\pm^0, U_\pm^1, 0, 0)$ the velocity vectors of the two counter-rotating fluids. In order to do the decomposition (3.1) we project the velocity vectors onto the tetrad $e_{\hat{a}}^b$, using the relations [24]

$$U_\pm^{\hat{a}} = e_{\hat{a}}^b U_\pm^b , \quad U_\pm^a = U_\pm^{\hat{c}} e_{\hat{c}}^a . \quad (3.2)$$

With the tetrad (2.8) we can write

$$U_{\pm}^a = \frac{V^a + U_{\pm} W^a}{\sqrt{1 - U_{\pm}^2}}, \quad (3.3)$$

and thus

$$V^a = \frac{\sqrt{1 - U_-^2} U_+ U_-^a - \sqrt{1 - U_+^2} U_- U_+^a}{U_+ - U_-}, \quad (3.4a)$$

$$W^a = \frac{\sqrt{1 - U_+^2} U_+^a - \sqrt{1 - U_-^2} U_-^a}{U_+ - U_-}, \quad (3.4b)$$

where $U_{\pm} = U_{\pm}^{\hat{1}}/U_{\pm}^{\hat{0}}$ are the tangential velocities of the fluids with respect to the tetrad.

In terms of the metric $h_{ab} = g_{ab} - Y_a Y_b$ of the hypersurface $z = 0$, we can write the SEMT as

$$S^{ab} = (\sigma + p_r) V^a V^b + (p_{\varphi} - p_r) W^a W^b + p_r h^{ab}, \quad (3.5)$$

and so, using (3.4), we obtain

$$\begin{aligned} S^{ab} &= \frac{f(U_-, U_-)(1 - U_+^2)}{(U_+ - U_-)^2} \frac{U_+^a U_+^b}{U_+^a U_+^b} \\ &+ \frac{f(U_+, U_+)(1 - U_-^2)}{(U_+ - U_-)^2} \frac{U_-^a U_-^b}{U_-^a U_-^b} \\ &- \frac{f(U_+, U_-)(1 - U_+^2)^{\frac{1}{2}}(1 - U_-^2)^{\frac{1}{2}}(U_+^a U_-^b + U_-^a U_+^b)}{(U_+ - U_-)^2} \\ &+ p_r h^{ab}, \end{aligned}$$

where

$$f(U_1, U_2) = (\sigma + p_r) U_1 U_2 + p_{\varphi} - p_r. \quad (3.6)$$

Thus, in order to cast the SEMT in the form (3.1), the mixed term must be absent and so the counter-rotating tangential velocities must be related by

$$f(U_+, U_-) = 0, \quad (3.7)$$

where we assume that $|U_{\pm}| \neq 1$.

Assuming a given choice for the counter-rotating velocities in agree with the above relation, we can write the SEMT as (3.1) with

$$S_{\pm}^{ab} = (\sigma_{\pm} + p_{\pm}) U_{\pm}^a U_{\pm}^b + p_{\pm} h^{ab}, \quad (3.8)$$

where

$$\sigma_+ + p_+ = \left[\frac{1 - U_+^2}{U_- - U_+} \right] \{(\sigma + p_r)U_-\}, \quad (3.9a)$$

$$\sigma_- + p_- = \left[\frac{1 - U_-^2}{U_+ - U_-} \right] \{(\sigma + p_r)U_+\}, \quad (3.9b)$$

$$\sigma_+ + \sigma_- = \sigma + p_r - p_{\varphi}, \quad (3.9c)$$

$$p_+ + p_- = p_r. \quad (3.9d)$$

Note that the counter-rotating energy densities σ_{\pm} and pressures p_{\pm} are not uniquely defined by the above relations, also for definite values of U_{\pm} .

Now we analyze the possibility of a complete determination of the vectors U_{\pm}^a . As we can see, the constraint (3.7) do not determines U_{\pm} uniquely, and so there is a freedom in the choice of U_{\pm}^a . The simplest, common, possibility is to take the two counter-rotating tangential velocities as equal and opposite; that is,

$$U_{\pm} = \pm U, \quad (3.10)$$

so that (3.7) is equivalent to

$$U^2 = \left[\frac{p_{\varphi} - p_r}{\sigma + p_r} \right]. \quad (3.11)$$

This choice, commonly considered, leads to a complete determination of the velocity vectors U_{\pm}^a ; however, this can be made only when the above expression is positive definite. If it is not the case, we will have a CRM valid only in a portion of the disk.

Another possibility, also commonly assumed, is to take the two counter-rotating fluids as circulating along geodesics. Let be $\omega_{\pm} = U_{\pm}^1/U_{\pm}^0$ the angular velocities obtained from the geodesic equation for a test particle,

$$g_{11,r}\omega^2 + g_{00,r} = 0, \quad (3.12)$$

so that

$$\omega_{\pm} = \pm \omega \quad , \quad \omega^2 = - \frac{g_{00,r}}{g_{11,r}} . \quad (3.13)$$

As the spacetime is static, the two geodesic fluids circulate with equal and opposite velocities and so this is a particular case of the above considered.

In order to see if the geodesic velocities agree with (3.7), we need to compute $f(U_+, U_-)$. In terms of ω_{\pm} we get

$$U_{\pm} = - \left[\frac{W_1}{V_0} \right] \omega_{\pm} , \quad (3.14)$$

and so, using (2.7a) and (2.7b), we can write

$$f(U_+, U_-) = \frac{A + (\sigma + p_r - p_{\varphi})B}{g_{11,r} V_0^2} , \quad (3.15)$$

where

$$A = g_{11,r} S_{00} + g_{00,r} S_{11} ,$$

$$B = g_{00} g_{11,r} + g_{00,r} g_{11} .$$

Using the Einstein equations (2.2a) - (2.2d) and the expressions (2.6a) - (2.6c) for the SEMT we can show that

$$f(U_+, U_-) = \left[\frac{\mathcal{R}}{\mathcal{R}_{,r} - \mathcal{R}\Phi_{,r}} \right] \frac{dp_r}{dr} ; \quad (3.16)$$

that is, the counter-rotating fluids circulate along geodesics only if the radial pressure is constant. In the general case, however, $f(U_+, U_-) \neq 0$ for fluids circulating along geodesics and so it is not possible to obtain a counter-rotating model with them.

As we can see of the above considerations, for disks built from generic static axially symmetric metrics, the counter-rotating velocities are not completely determined by the constraint (3.7). Thus, the CRM is in general undetermined since the energy density and isotropic pressure can not be explicitly written without a knowledge of the counter-rotating tangential velocities.

4 A Family of Disks with Some Stable CRM

We will now consider a simple specific example where we can obtain some CRM with well defined counter-rotating velocities. In order to obtain finite

static disks with nonzero radial pressure we consider, following reference [5], a solution of (2.2a) - (2.2d) obtained by taking

$$\mathcal{F}(\nu) = \nu + \alpha\sqrt{\nu^2 - 1} , \quad (4.1)$$

where $\alpha \geq 0$, so representing a thin disk located at $z = 0$, $0 \leq r \leq 1$. Now we take a simple solution of Weyl equations (2.4a) - (2.4c) given by [25]

$$\Psi(\mathcal{R}, \mathcal{Z}) = \frac{\mu}{2k} \ln \left[\frac{\mathcal{R}_+ + \mathcal{R}_- - 2k}{\mathcal{R}_+ + \mathcal{R}_- + 2k} \right] , \quad (4.2a)$$

$$\Pi(\mathcal{R}, \mathcal{Z}) = \frac{\mu^2}{2k^2} \ln \left[\frac{(\mathcal{R}_+ + \mathcal{R}_-)^2 - 4k^2}{4 \mathcal{R}_+ \mathcal{R}_-} \right] , \quad (4.2b)$$

where $\mu > 0$, $k = \sqrt{\alpha^2 - 1}$ and $\mathcal{R}_\pm^2 = \mathcal{R}^2 + (\mathcal{Z} \pm k)^2$.

From the above expressions, and using (2.6a) - (2.6c), we can compute the energy density and azimuthal and radial pressures of the disks. We obtain

$$\sigma = p_r \left[\frac{2\mu - \alpha}{\alpha} - \frac{1 + \mu^2 r^2}{1 + k^2 r^2} \right] , \quad (4.3a)$$

$$p_\varphi = p_r \left[\frac{1 + \mu^2 r^2}{1 + k^2 r^2} \right] , \quad (4.3b)$$

$$p_r = p_0 [1 + k^2 r^2]^{(\mu^2 - k^2)/2k^2} , \quad (4.3c)$$

where $p_0 = 2\alpha(\alpha - k)^{\mu/k} \geq 0$ and $0 \leq r \leq 1$.

We consider, in the first instance, two cases where we obtain simple expressions for the SEMT. Let be $\alpha = 0$, so that $p_r = p_\varphi = 0$, and

$$\sigma = \frac{4\mu e^{\mu\pi/2}}{(1 - r^2)^{(\mu^2+1)/2}} . \quad (4.4)$$

We have a disk of dust with positive energy density, so in agree with all the energy conditions [26]. On the other hand, the energy density is singular at the edge of the disk. We can also see that the constraint (3.7) leads to $U_\pm = 0$, so this is a “true static disk”, in the sense that we can not obtain a counter-rotating interpretation for it. This case corresponds to the Bonnor and Sackfield disk of reference [1]. For the second simple case, we take $\mu = k > 0$, so that $p_r = p_\varphi = 2\alpha(\alpha - k)$ and

$$\sigma = -4(\alpha - k)^2 . \quad (4.5)$$

The disks so are made of perfect fluids with constant energy density and pressure. As $\sigma < 0$, the disks do not agree with the weak energy condition, but $\varrho > 0$, as the strong energy condition requires. For these disks we also have that $U_{\pm} = 0$, so that we do not have a CRM. These are also “true static disks”.

For any other value of $\alpha > 0$ and $\mu \neq k$, the radial and azimuthal pressures, p_r and p_{φ} , are everywhere positive and well behaved for all the values of μ and α . For the effective Newtonian density we obtain

$$\varrho = \frac{2\mu p_r}{\alpha}, \quad (4.6)$$

so that is positive everywhere on the disk. Thus, the disks are attractive, in agreement with the strong energy condition. On the other hand, it is easy to see that $\sigma < 0$ when $r = 1$, for any value of α and μ , whereas that $\sigma > 0$ at $r = 0$ only if $\mu > \alpha$. That is, in general, the energy density σ is not positive everywhere on the disk.

We now consider the CRM for the above disks. As p_r is, in general, dependent of r , we can not take the two counter-rotating fluids as circulating along geodesics. However, we can test the possibility of obtain a well defined CRM with equal and opposite velocities. In order to do this, we can compute, from (3.11) and (4.3), the tangential velocity U^2 and obtain

$$U^2 = \frac{\alpha(\mu^2 - k^2)r^2}{(2\mu - \alpha) + \mu(2k^2 - \alpha\mu)r^2}, \quad (4.7)$$

and, in order to have a well behaved CRM, we impose the condition

$$0 \leq U^2 \leq 1. \quad (4.8)$$

We study the above relation for a lot of combinations of the parameters μ and α . In many of the cases we obtain functions with strong change in the slope, with regions where U^2 is negative and with $U > 1$. However, in some other cases we obtain U^2 as well behaved functions of r , everywhere positive and increasing, but always with $U^2 < 1$.

Finally, we can compute $\sigma_+ + \sigma_-$ and $\sigma_{\pm} + p_{\pm}$ for the above disks. Using (3.9), (3.11) and (4.3), we obtain

$$\sigma_+ + \sigma_- = 2 p_r \left[\frac{\mu}{\alpha} - \frac{1 + \mu^2 r^2}{1 + k^2 r^2} \right], \quad (4.9a)$$

$$\sigma_{\pm} + p_{\pm} = p_r \left[\frac{2\mu + \alpha}{2\alpha} - \frac{1 + \mu^2 r^2}{1 + k^2 r^2} \right]. \quad (4.9b)$$

We study the above relations for the values of the parameters μ and α that leads to well behaved tangential velocities and specific angular momentum. It is easy to see that, for these values of the parameters, the total energy density of the CRM, $\sigma_+ + \sigma_-$, is positive only in the central region of the disks but it is negative in the rest, having a maximum at $r = 0$ and then decrease monotonly. On the other hand, $\sigma_{\pm} + p_{\pm}$ is always positive for these cases. We see that the CRM obtained are agree with the strong energy condition, although not with the weak energy condition.

5 Discussion

We presented a detailed study of the Counter-Rotating Model for generic finite static axially symmetric thin disks, with nonzero radial pressure. A general constraint over the counter-rotating tangential velocities was obtained, needed to cast the surface energy-momentum tensor of the disk in such a way that can be interpreted as the superposition of two counter-rotating perfect fluids. The constraint obtained is the generalization of the obtained in [18], for disks without radial pressure or heat flow, where we only consider counter-rotating fluids circulating along geodesics. Also, we obtain expressions for the energy density and pressure of the counter-rotating fluids in terms of the energy density and azimuthal and radial pressures of the disk.

We shown that, in general, there is not possible to take the two counter-rotating tangential velocities as equal and opposite neither take the two counter-rotating fluids as circulating along geodesics. Thus, for disks built from generic static axially symmetric metrics, the counter-rotating velocities are not completely determined; that is, the CRM is in general undetermined since the energy density and isotropic pressure can not be explicitly written without a knowledge of the counter-rotating tangential velocities.

An specific example was considered of a family of disks where we obtain some stable CRM with well defined counter-rotating tangential velocities and agree with the strong energy condition, but with regions of the disks where the energy density is negative, so in violation of the weak energy condition. We also found some disks of the family with negative U^2 so that was not possible to obtain CRM for these disks. Were found also two cases of “true static disks”, in the sense of $U_{\pm} = 0$, and so there is not a possible CRM interpretation.

The generalization of the Counter-Rotating Model presented here to the case of rotating thin disks with or without radial pressure is in consideration. Also, the generalization for static and stationary disks with magnetic or electric

fields is being considered.

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